Session L:

Modeling Change Using Latent Variable Factor Analysis Models

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Overview

1. Structural Basis of the Common Factor Model
2. An Example of Common Factor Modeling
3. Factorial Invariance over Time
4. An Example of Common Factor Invariance over Time
5. Modeling Change over Time at the Latent Variable Level
6. Summary & Discussion

A longitudinal common factor model from Jöreskog & Sörbom (1975)

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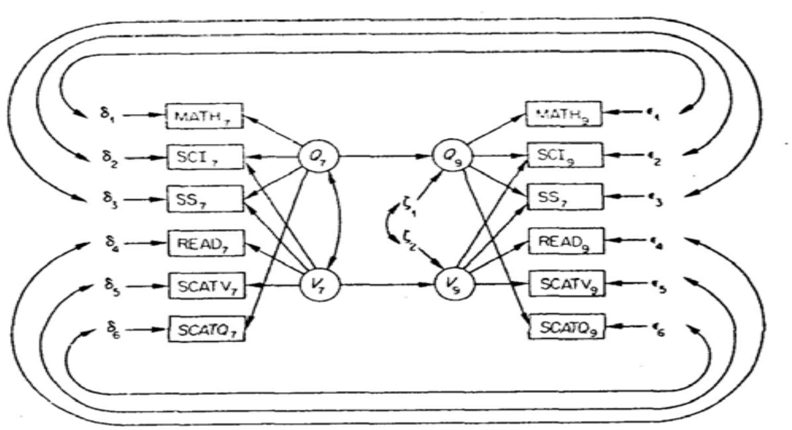


Figure 3. Revised model for the measurement of change in verbal and quantitative ability between grades 7 and 9

1. Structural Basis of the Common Factor Model

Basic definitions of "factor analysis"

- Factor analysis
  - computational techniques widely use in research on individual differences
  - a mathematical model used to express observations in terms of latent variables
- Factors
  - theoretical / hypothetical constructs
  - enable testing rejectable hypotheses about empirical data (i.e., enable ***science*** of individual differences)
  - are "functional unities" that may be correlated (after Thurstone, 1947; Cattell, 1966; Horn, 1972)

Exploratory Factor Analysis (EFA)

- Much can be learned by first using standard techniques in exploratory factor analysis (EFA).
- EFA may find any initial problems with data, and this is often the case.
- EFA can be used to establish a boundary conditions about likely the number of common factors, and this can save a lot of time later.
- Advanced techniques in EFA can be so close to the final restrictions desired that no further analyses may be necessary.

Structural Factor Analysis

- Standard techniques in *confirmatory factor analysis* (CFA) are also often very informative.
- If explicit point hypotheses not stated a priori, we use term *structural factor analysis* (SFA).
- SFA analyses can test explicit hypotheses about the likely number of common factors, with precise estimation and resulting statistical tests.
- SFA can test precise restrictions in a longitudinal model – including unique factor covariances and invariance of loadings, intercepts, etc.

Structural Factor Analysis (SFA) Expectations

- Factor models imply expectations in means and covariances
- Factor models can be distinguished by their patterned expectations
- Figural models are isomorphic with (usually linear) factor model representations of variables
- A little matrix algebra and expectation operators can make clear distinctions among models
- Assume the presence of 4 manifest variables …

Fundamental Equation of EFA, SFA

- Linear model underlying factor analysis

$$Y_{ji} = \tau_j + \lambda_{j1} f_{1i} + \lambda_{j2} f_{2i} + \varepsilon_{ji}$$

- where
  - Y_{ji} = score of person i on manifest variable *j*
  - τ_j = intercept for manifest variable *j*
  - λ_{j1} , λ_{j2} = loading of variable *j* on Factors 1 & 2
  - f_{1i} , f_{2i} = scores of person *i* on Factors 1 & 2
  - ε_{ji} = score of person *i* on unique factor for manifest variable *j*

SFA Expectations: 1 Factor

- Collect linear model for 4 manifest variables, 1 factor

$$\begin{bmatrix} Y_{1i} \\ Y_{2i} \\ Y_{3i} \\ Y_{4i} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix} + \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \\ \lambda_{41} \end{bmatrix} \begin{bmatrix} f_{1i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \end{bmatrix}$$

- Rewrite compactly $\quad \mathbf{Y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\mathbf{f} + \boldsymbol{\varepsilon}$

- Take expectations to represent means

$$E(\mathbf{Y}) = E(\boldsymbol{\tau} + \boldsymbol{\Lambda}\mathbf{f} + \boldsymbol{\varepsilon}) = \boldsymbol{\tau} + \boldsymbol{\Lambda}\, E(\mathbf{f}) + E(\boldsymbol{\varepsilon})$$
$$= \boldsymbol{\tau} + \boldsymbol{\Lambda}\,\boldsymbol{\alpha} + 0 = \qquad \boldsymbol{\tau} + \boldsymbol{\Lambda}\,\boldsymbol{\alpha} = \boldsymbol{\mu}$$

SFA Covariance Expectations

- If **y** is vector of manifest variables in mean deviation form, taking expectations for covariances we get:

$$E(\mathbf{y}\mathbf{y}') =$$
$$\boldsymbol{\Sigma} = E\left[(\boldsymbol{\Lambda}\mathbf{f} + \boldsymbol{\varepsilon})(\boldsymbol{\Lambda}\mathbf{f} + \boldsymbol{\varepsilon})'\right] = E\left[(\boldsymbol{\Lambda}\mathbf{f} + \boldsymbol{\varepsilon})(\mathbf{f}'\boldsymbol{\Lambda}' + \boldsymbol{\varepsilon}')\right] =$$
$$\boldsymbol{\Sigma} = E(\boldsymbol{\Lambda}\mathbf{f}\mathbf{f}'\boldsymbol{\Lambda}' + \boldsymbol{\Lambda}\mathbf{f}\boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}\mathbf{f}'\boldsymbol{\Lambda}' + \boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')$$
$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}E(\mathbf{f}\mathbf{f}')\boldsymbol{\Lambda}' + \boldsymbol{\Lambda}E(\mathbf{f}\boldsymbol{\varepsilon}') + E(\boldsymbol{\varepsilon}\mathbf{f}')\boldsymbol{\Lambda}' + E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')$$
$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}$$

SFA Covariance Expectations: 1 Factor

- Given the above, what are expectations for covariances?

$$\boldsymbol{\Sigma} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \begin{bmatrix} \psi_{11} \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix} + \begin{bmatrix} \theta_{11} & & & \text{symm} \\ 0 & \theta_{22} & & \\ 0 & 0 & \theta_{33} & \\ 0 & 0 & 0 & \theta_{44} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\psi_{11} \\ \lambda_2\psi_{11} \\ \lambda_3\psi_{11} \\ \lambda_4\psi_{11} \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix} + \begin{bmatrix} \theta_{11} & & & \text{symm} \\ 0 & \theta_{22} & & \\ 0 & 0 & \theta_{33} & \\ 0 & 0 & 0 & \theta_{44} \end{bmatrix}$$

SFA Covariance Expectations: 1 Factor

$$= \begin{bmatrix} \lambda_1^2\psi_{11} & & & \text{symm} \\ \lambda_2\lambda_1\psi_{11} & \lambda_2^2\psi_{11} & & \\ \lambda_3\lambda_1\psi_{11} & \lambda_3\lambda_2\psi_{11} & \lambda_3^2\psi_{11} & \\ \lambda_4\lambda_1\psi_{11} & \lambda_4\lambda_2\psi_{11} & \lambda_4\lambda_3\psi_{11} & \lambda_4^2\psi_{11} \end{bmatrix} + \begin{bmatrix} \theta_{11} & & & \text{symm} \\ 0 & \theta_{22} & & \\ 0 & 0 & \theta_{33} & \\ 0 & 0 & 0 & \theta_{44} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^2\psi_{11}+\theta_{11} & & & \text{symm} \\ \lambda_2\lambda_1\psi_{11} & \lambda_2^2\psi_{11}+\theta_{22} & & \\ \lambda_3\lambda_1\psi_{11} & \lambda_3\lambda_2\psi_{11} & \lambda_3^2\psi_{11}+\theta_{33} & \\ \lambda_4\lambda_1\psi_{11} & \lambda_4\lambda_2\psi_{11} & \lambda_4\lambda_3\psi_{11} & \lambda_4^2\psi_{11}+\theta_{44} \end{bmatrix}$$

SFA Covariance Expectations: General

- Additional constraints are needed to obtain a *unique* set of estimates. The latent variable has no meaningful absolute scale, so we assume it has a mean of 0.0 and variance $E\{\mathbf{f}, \mathbf{f}\}=\sigma_f^2=1.0$
- After this scaling, each covariance is simply a product of the pairs of loadings:

  $$E\{\sigma(i,j)\} = \lambda(i) \,.\, \lambda(j)$$
- We could achieve identification by setting the factor variance $E\{\mathbf{f}, \mathbf{f}\}$=1/2 or 25 or any positive constant, or by setting at least one loading equal to some constant ($\lambda(i)$ =1). Different choices alter the estimates, but not the fit of the model to data.

SFA Covariance Expectations: 1 Factor

- So, if ψ_{11} is fixed at 1.0 for identification

$$= \begin{bmatrix} \lambda_1^2+\theta_{11} & & & \text{symm} \\ \lambda_2\lambda_1 & \lambda_2^2+\theta_{22} & & \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2+\theta_{33} & \\ \lambda_4\lambda_1 & \lambda_4\lambda_2 & \lambda_4\lambda_3 & \lambda_4^2+\theta_{44} \end{bmatrix}$$

Note: Each covariance σ_{ij} expectation follow a simple pattern determined by the product of the respective loadings λ_i and λ_j.

SFA Covariance Expectations: 2 Factors

- What are expectations for covariances based on 2 factors?

$$\mathbf{\Sigma} = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & 0 \\ 0 & \lambda_3 \\ 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} 1.0 & \psi_{12} \\ \psi_{21} & 1.0 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{bmatrix} + \begin{bmatrix} \theta_{11} & & & \text{symm} \\ 0 & \theta_{22} & & \\ 0 & 0 & \theta_{33} & \\ 0 & 0 & 0 & \theta_{44} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \lambda_1\psi_{12} \\ \lambda_2 & \lambda_2\psi_{12} \\ \lambda_3\psi_{21} & \lambda_3 \\ \lambda_4\psi_{21} & \lambda_4 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{bmatrix} + \begin{bmatrix} \theta_{11} & & & \text{symm} \\ 0 & \theta_{22} & & \\ 0 & 0 & \theta_{33} & \\ 0 & 0 & 0 & \theta_{44} \end{bmatrix}$$

SFA Covariance Expectations: 2 Factors

$$
= \begin{bmatrix} \lambda_1^2 & & & \text{symm} \\ \lambda_2\lambda_1 & \lambda_2^2 & & \\ \lambda_3\psi_{21}\lambda_1 & \lambda_3\psi_{21}\lambda_2 & \lambda_3^2 & \\ \lambda_4\psi_{21}\lambda_1 & \lambda_4\psi_{21}\lambda_2 & \lambda_4\lambda_3 & \lambda_4^2 \end{bmatrix} + \begin{bmatrix} \theta_{11} & & & \text{symm} \\ 0 & \theta_{22} & & \\ 0 & 0 & \theta_{33} & \\ 0 & 0 & 0 & \theta_{44} \end{bmatrix}
$$

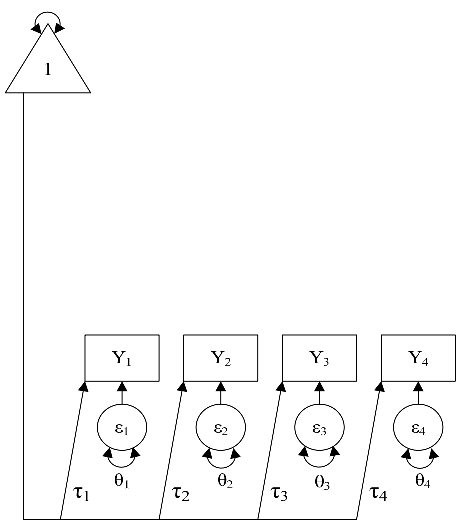
$$
= \begin{bmatrix} \lambda_1^2+\theta_{11} & & & \text{symm} \\ \lambda_2\lambda_1 & \lambda_2^2+\theta_{22} & & \\ \lambda_3\psi_{21}\lambda_1 & \lambda_3\psi_{21}\lambda_2 & \lambda_3^2+\theta_{33} & \\ \lambda_4\psi_{21}\lambda_1 & \lambda_4\psi_{21}\lambda_2 & \lambda_4\lambda_3 & \lambda_4^2+\theta_{44} \end{bmatrix}
$$

Same as covariance expectations for 1 factor model – except addition of ψ_{21} to four covariances. If ψ_{21} nears 1.0, becomes hard to tell 1 factor model from 2 factor model

SFA Models

- Having derived mean and covariance expectations, we can now consider 3 alternative models that might fit covariances among 4 manifest variables:
  - 0 factor model
  - 1 factor model
  - 2 factor restricted model

A path diagram of a no factor model



14 statistics –
8 estimates (or
4 intercepts +
4 unique vars)
= 6 *df*

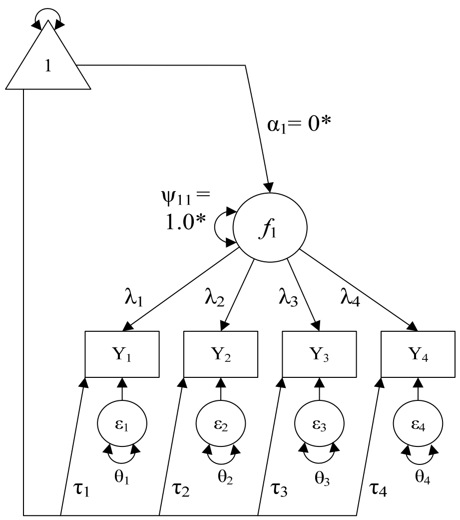
SFA Covariance Expectations: 0 Factor

- All factor loadings are 0, so 0 factor baseline model has following expectations

$$
\mathbf{\Sigma} = \begin{bmatrix} \theta_{11} & & & \text{symm} \\ 0 & \theta_{22} & & \\ 0 & 0 & \theta_{33} & \\ 0 & 0 & 0 & \theta_{44} \end{bmatrix}
$$

Note: Reproduces variance of each manifest variable, but covariances of 0 between manifest variables because common factor is absent

A path diagram of a one factor model



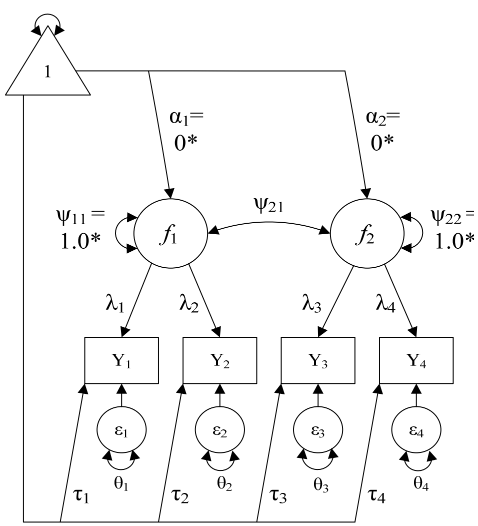
14 statistics –
12 estimates (or
4 intercepts +
4 loadings +
4 unique vars)
= 2 *df*

SFA Covariance Expectations: 1 Factor

$$\boldsymbol{\Sigma} = \begin{bmatrix} \lambda_1^2 + \theta_{11} & & & \text{symm} \\ \lambda_2\lambda_1 & \lambda_2^2 + \theta_{22} & & \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2 + \theta_{33} & \\ \lambda_4\lambda_1 & \lambda_4\lambda_2 & \lambda_4\lambda_3 & \lambda_4^2 + \theta_{44} \end{bmatrix}$$

Note: Each covariance σ_{ij} expectation follow a simple pattern determined by the product of the respective loadings λ_i and λ_j.

A path diagram of two factor model



14 statistics –
13 estimates (or
4 intercepts +
4 loadings +
4 unique vars +
1 factor corr)
= 1 *df*

SFA Covariance Expectations: 2 Factors

$$\boldsymbol{\Sigma} = \begin{bmatrix} \lambda_1^2 + \theta_{11} & & & \text{symm} \\ \lambda_2\lambda_1 & \lambda_2^2 + \theta_{22} & & \\ \lambda_3\psi_{21}\lambda_1 & \lambda_3\psi_{21}\lambda_2 & \lambda_3^2 + \theta_{33} & \\ \lambda_4\psi_{21}\lambda_1 & \lambda_4\psi_{21}\lambda_2 & \lambda_4\lambda_3 & \lambda_4^2 + \theta_{44} \end{bmatrix}$$

Note: Again note extra component of covariance expectations for the "between factor" covariances

Goodness of Fit of a Common Factor Model

- Many tests/indices of fit (Lawley & Maxwell, 1971; Browne, 1985; Browne & Cudeck, 1993).
- If we calculate the parameters based on the principle of *maximum likelihood estimation* (MLE) we obtain a *likelihood ratio statistic* (L^2) of "misfit."
- Under standard assumptions (e.g., normality of the residuals) L^2 follows a χ^2 distribution with df=Ns-Np.
- We use L^2 type tests to ask: "Should we reject the hypothesis of local independence | *k*-common factors."
- Often we rephrase this: "Are the observed data *consistent* with the hypothetical model?"or "Is the model plausible?" Or simply: "Does the model fit?"

Confirmatory Factor Analysis Hypotheses

- If restrictions are stated prior to estimation then we can use statistical probability models.
- These models have large numbers of degrees-of-freedom and are rejectable -- we examine overall fit, standard errors, and residuals.
- We need to examine most restrictions via the comparison of *at least two alternative models*.
- We do not conclude the model fits the data, but we can conclude the model does not fit the data, or that one model fits better than another.
- We might be better off considering a wide range of models and judge the relative fit of each.

Nested models and difference tests

- The well-known use of a hierarchy of restrictions makes it easy to use statistical probability tests.
- Two alternative models are said to be "nested" if parameters that are included in one model (M_2) can be are "removed" to form the alternative model (M_1).
- *Iff* Model 1 is nested in Model 2, we can use standard statistical difference tests →

$$\Delta L^2 = L^2(1) - L^2(2)$$

follows a *chi-square* distribution with

$$\Delta df = df(1) - df(2)$$

Parsimony versus Good Fit

- "Which model(s) should be considered a reasonable fit to data?" – This is a well-known, complex question.
- Want parsimonious model with few parameter estimates
- Fit indices help us do this:
  - Likelihood ratio chi-square
  - Information indices: Akaike IC (AIC), Bayesian IC (BIC is Better)
  - Relative fit indices: Relative Noncentrality Index (RNI) (also called Comparative Fit Index (CFI), and the Tucker-Lewis Index (TLI)
  - Absolute fit indices: RMSEA & its confidence interval
  - Prefer indices with correction for model complexity – especially BIC, TLI, and RMSEA

Relative Goodness of Fit in Factor Analysis

- Goodness-of-fit is relative to data at hand -- comparisons *within* a data set are informative.
- Absolute rules for goodness-of-fit indices are useful (but wrong) – i.e., “non-significant χ^2,” or “smallest AIC” or “*RMSEA* < .05”!
- “Factors in a factor analysis are not *things*, but they are our evidence for the *existence of things*.” (R.B. Cattell, 1950).
- Substantive knowledge is always needed.

2: Two Examples of Common Factor Modeling

From McArdle & Nesselroade (1994)

- WISC-R data on *N*=204 children repeatedly measured at ages 6 and 11.
- Variables chosen for analysis are all from the “Verbal” scales of the WISC at age 6
  - 1. Information
  - 2. Similarities
  - 3. Comprehension
  - 4. Vocabulary
- Common factor models were fitted to test various factor analytic hypotheses.

WISC sample statistics (age 6)

Pearson Correlation Coefficients, N = 204

	info_06	voca_06	comp_06	simi_06
info_06	1.000			
voca_06	0.556	1.000		
comp_06	0.509	0.584	1.000	
simi_06	0.540	0.437	0.449	1.000
MEAN	19.776	20.396	21.797	14.903
STD	6.119	6.292	9.742	7.56

Mplus program: 1 factor model

```
TITLE:    V subtests: 1 Factor model
DATA:     FILE = rto8age2.dat;
VARIABLE: NAMES =  ID moeducat age_06 info_06 comp_06 simi_06 voca_06
          picc_06 pica_06 bloc_06 obje_06  age_11 info_11 comp_11
          simi_11 voca_11 picc_11 pica_11 bloc_11 obje_11;
          USEVAR =
          info_06 voca_06 comp_06 simi_06 ;

ANALYSIS: TYPE=MEANSTRUCTURE; ITERATIONS=10000;
MODEL:   !Loadings of Manifest on Latent Variables (L = loadings)
          Vc_06 BY info_06 *(L1)
                   voca_06  (L2)
                   comp_06  (L3)
                   simi_06  (L4) ;
         !Intercepts (or taus) of Manifest Variables (I = intercepts)
          [info_06 voca_06 comp_06 simi_06];
         !Unique variances and covariances (U = intercepts)
          info_06 voca_06 comp_06 simi_06 ;
         !Latent variable means (A = alpha)
          [Vc_06@0];
         !Latent variable variances and covariances (P = psi)
           Vc_06@1;
         !Latent variable regressions (B = beta)

OUTPUT:   SAMPSTAT STANDARDIZED RESIDUAL tech1;
```


M+ output of common factor model

```
TESTS OF MODEL FIT
Chi-Square Test of Model Fit
          Value                           7.147
          Degrees of Freedom                  2
          P-Value                        0.0274 ← P{perfect fit}
Chi-Square Test of Model Fit for the Baseline Model
          Value                         261.106
          Degrees of Freedom                  6
          P-Value                        0.0000
CFI/TLI   CFI                             0.980
          TLI                             0.939
Loglikelihood
          H0 Value                    -2650.669
          H1 Value                    -2647.095
Information Criteria
          Number of Free Parameters          12
          Akaike (AIC)                 5325.338
          Bayesian (BIC)               5365.155
                 ...
RMSEA (Root Mean Square Error Of Approximation)
          Estimate                        0.112
          90 Percent C.I.                 0.032  0.206
          Probability RMSEA <= .05        0.088 ← P{close fit}
SRMR (Standardized Root Mean Square Residual)
          Value                           0.025
```


Misfit of common factor model

```
ESTIMATED MODEL AND RESIDUALS (OBSERVED - ESTIMATED)
           Model Estimated Means/Intercepts/Thresholds
              INFO6         COMP6         SIMI6         VOCA6
              ________      ________      ________      ________
      1        19.776        21.797        14.903        20.396
           Residuals for Means/Intercepts/Thresholds
              INFO6         COMP6         SIMI6         VOCA6
              ________      ________      ________      ________
      1         0.000         0.000         0.000         0.000

           Model Estimated Covariances/Correlations/Residual Correlations
              INFO6         COMP6         SIMI6         VOCA6
              ________      ________      ________      ________
 INFO6         37.261
 COMP6         32.128        94.434
 SIMI6         22.189        33.963        56.881
 VOCA6         21.598        33.058        22.831        39.390

           Residuals for Covariances/Correlations/Residual Correlations
              INFO6         COMP6         SIMI6         VOCA6
              ________      ________      ________      ________
 INFO6          0.000
 COMP6         -1.952         0.000
 SIMI6          2.680        -1.076         0.000
 VOCA6         -0.313         2.569        -2.140         0.000
```

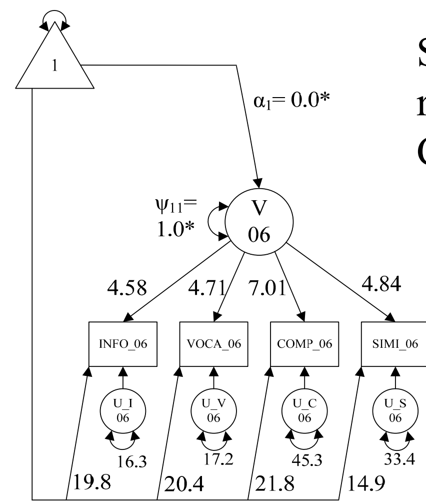

M+ output of common factor model

```
MODEL RESULTS                                    Two-Tailed    Standard
                    Estimate       S.E.  Est./S.E.    P-Value    Estimate
 VC_06    BY
    INFO_06            4.582      0.412     11.112      0.000       0.751
    VOCA_06            4.714      0.421     11.192      0.000       0.751
    COMP_06            7.013      0.659     10.642      0.000       0.722
    SIMI_06            4.843      0.527      9.190      0.000       0.642

 Means
    VC_06              0.000      0.000    999.000    999.000       0.000

 Intercepts
    INFO_06           19.776      0.427     46.273      0.000       3.240
    VOCA_06           20.396      0.439     46.416      0.000       3.250
    COMP_06           21.797      0.680     32.036      0.000       2.243
    SIMI_06           14.903      0.528     28.224      0.000       1.976

 Variances
    VC_06              1.000      0.000    999.000    999.000       1.000

 Residual Variances
    INFO_06           16.271      2.419      6.725      0.000       0.437
    VOCA_06           17.167      2.517      6.820      0.000       0.436
    COMP_06           45.258      6.176      7.328      0.000       0.479
    SIMI_06           33.426      4.030      8.293      0.000       0.588
```

Alternative Factor Models for Verbal Tests

WISC Verbal Tests	Tests	0 Factors	1 Factor Raw λ [std λ]	2 Factors Raw λ [std λ]
Factor Loadings on Factor 1 (Vc)	IN VO CO SI	--- --- --- ---	4.58 [.75] 4.71 [.75] 7.01 [.72] 4.84 [.64]	4.55 [.75] 4.68 [.75] 0.00* [.00] 0.00* [.00]
Factor Loadings on Factor 2 (Vr)	IN VO CO SI	--- --- --- ---	--- --- --- ---	0.00* [.00] 0.00* [.00] 6.88 [.71] 4.78 [.63]
Factor 1 Var Factor 2 Var		--- ---	1.00* ----	1.00* 1.00*
Factor covariance		---	----	1.04
χ^2 / *df* fit		261.1 / 6	7.2 / 2	6.8 / 1
χ^2 / *df* change		---	253.9 / 4	0.4 / 1
CFI / TLI		.000/.000	.980/.939	.977/.863
RMSEA(CI)		.457 (.410,.505)	.112 (.032,.206)	.169 (.067,.298)

Optimal Model WISC Verbal: 1 factor



Single common factor – represents Verbal Comprehension (or Gc)

Example 2: Gc-Gf example

- Same WISC-R data on *N*=204 children repeatedly measured at ages 6 and 11.
- Variables chosen for analysis from age 6, but 2 Verbal tests and 2 Performance tests
  - 1. Comprehension (Vc or Gc)
  - 2. Vocabulary (Vc or Gc)
  - 3. Block Design (P or Gf)
  - 4. Object Assembly (P or Gf)
- Common factor models were fitted to test various factor analytic hypotheses.

WISC sample statistics (age 6) – Gc-Gf example

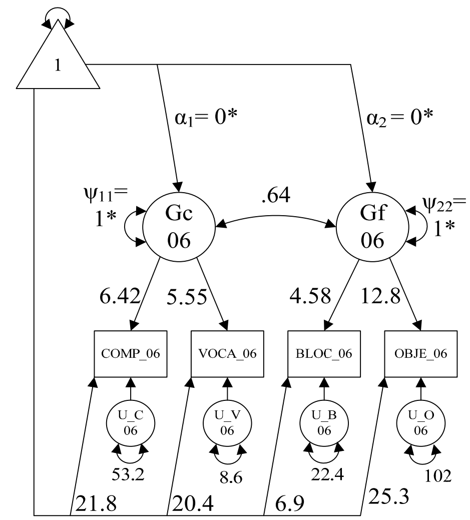
Pearson Correlation Coefficients, N = 204

	comp_06	voca_06	bloc_06	obje_06
comp_06	1.000			
voca_06	0.584	1.000		
bloc_06	0.324	0.385	1.000	
obje_06	0.313	0.450	0.545	1.000
MEAN	21.797	20.396	6.942	25.258
STD	9.742	6.292	6.599	16.395

Alternative Factor Models for Gc-Gf Test

WISC Verbal Tests	Tests	0 Factors	1 Factor Raw λ [std λ]	2 Factors Raw λ [std λ]
Factor Loadings on Factor 1 (Gc)	CO	---	6.41 [.66]	6.42 [.66]
	VO	---	4.89 [.78]	5.55 [.88]
	BL	---	3.79 [.56]	0.00* [.00]
	OB	---	10.1 [.62]	0.00* [.00]
Factor Loadings on Factor 2 (Gf)	CO	---	---	0.00* [.00]
	VO	---	---	0.00* [.00]
	BL	---	---	4.58 [.70]
	OB	---	---	12.8 [.78]
Factor 1 Var		---	1.00*	1.00*
Factor 2 Var		---	----	1.00*
Factor covariance		---	----	.64
χ^2 / *df* fit		214.2 / 6	31.5 / 2	1.2 / 1
χ^2 / *df* change		---	182.7: 4	30.3 / 1
CFI / TLI		.000/.000	.859/.576	.999/.995
RMSEA(CI)		.412 (.366,.461)	.269 (.191,.355)	.029 (.000,.191)

Optimal Model for Gf-Gc model: 2 factors



2 factors required – Key parameter is correlation between factors of .64 (SE = .07), Differs from 0, Differs from 1!

3. Concepts of factorial invariance over time

Factorial Invariance under Selection

- In a seminal series of papers, Meredith's (1964-65) extended Lawley's (1941) selection theorems to the common factor case and demonstrated:

  IF (1) a factor model $\Sigma = \Lambda\Phi\Lambda' + \Psi^2$ holds in a population,
  (2) samples are selected from that population in any way, randomly or non-randomly,

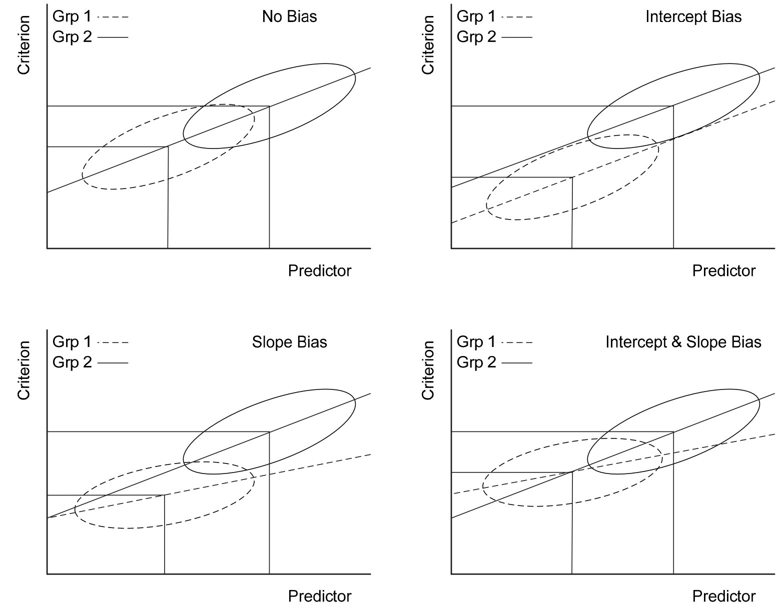
  THEN (3) the factor loadings Λ will remain invariant,
  (4) but the factor variances and covariances Φ will not remain invariant.

- Meredith (1993) extended this to consideration of intercepts – and related factorial invariance to measurement invariance more generally

Factorial Invariance

- Related to measurement invariance
- Test bias models (Cleary models), for G groups
  - Test unbiased *iff* $E(Y|X,G) = E(Y|X)$
  - $E(Y|X,G) = E(Y|X)$ *iff* intercept (a) and slope (b) of regression of criterion on predictor are equal across groups, or across G groups
  - Intercept: $a_1 = a_2 = \ldots = a_G = a$
  - Slope: $b_1 = b_2 = \ldots = b_G = b$
  - Different forms of bias can be represented as …

Test Bias Models



Factorial Invariance across Time

- In factorial invariance, manifest variables ***Y*** are criteria, latent variables ***f*** are predictors
- Thus, in factorial invariance, across T times
  - Factors are identical *iff* $E(\mathbf{Y}|\mathbf{f},T) = E(\mathbf{Y}|\mathbf{f})$
  - $E(\mathbf{Y}|\mathbf{f},T) = E(\mathbf{Y}|\mathbf{f})$ *iff* intercepts ($\boldsymbol{\tau}$) and slopes ($\boldsymbol{\Lambda}$) of regression of manifest variables on latent variables are equal across T times of measurement
  - Intercepts: $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \ldots = \boldsymbol{\tau}_G = \boldsymbol{\tau}$
  - Slopes: $\boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_2 = \ldots = \boldsymbol{\Lambda}_G = \boldsymbol{\Lambda}$
- Invariance of unique variances would ensure entire function relating latent to manifest variables is invariant

Longitudinal Change SEM

- Long history of longitudinal change models.
- Included are seminal SEM work by Jöreskog (1971, 1974, 1977) & Sörbom (1975)
- Related to factor analytic change models of Nesselroade (1971, 1977) & Meredith (1990, 1991; Meredith & Tisak, 1990)
- Widaman & Reise (1997) extended prior work on factorial invariance, outlining 4 levels of invariance

Types of Factor Invariance (FI)

- **Configural invariance:** same number of factors and same pattern of fixed and free loadings on factors at all times of measurement
- **Weak FI:** Configural invariance plus invariance of factor loadings across time – invariant $\mathbf{\Lambda}$
- **Strong FI:** Weak FI plus invariance of intercepts across time, or invariant $\mathbf{\Lambda}$ and $\boldsymbol{\tau}$
- **Strict FI:** Strong FI plus invariance of unique variances across time, or invariant $\mathbf{\Lambda}$, $\boldsymbol{\tau}$, and $\mathbf{\Theta}$

Types of Factor Invariance (FI)

- **Strong FI:** If Strong FI holds, same latent variables are found at all times of measurement; changes in mean and variance on latent variables (i.e., true scores) can be investigated
- **Strict FI:** If Strict FI holds, all change in mean and variance on manifest variables is due to change in mean and variance on latent variables
- **Important:** We must achieve strong FI to study change in the same factor(s) across time, and strict FI is certainly desirable (but not necessary)

Testing Factor Invariance (FI)

- **Configural invariance:** baseline model
- **Weak FI:** Compare to configural invariance model to test invariance of factor loadings across time – invariant $\mathbf{\Lambda}$
- **Strong FI:** Compare to Weak FI model to test invariance of intercepts across time, or invariant $\mathbf{\Lambda}$ and $\boldsymbol{\tau}$
- **Strict factorial invariance:** Compare to Strong FI model to test invariance of unique variances across time, or invariant $\mathbf{\Lambda}$, $\boldsymbol{\tau}$, and $\mathbf{\Theta}$

Do NOT Separately Standardize Data

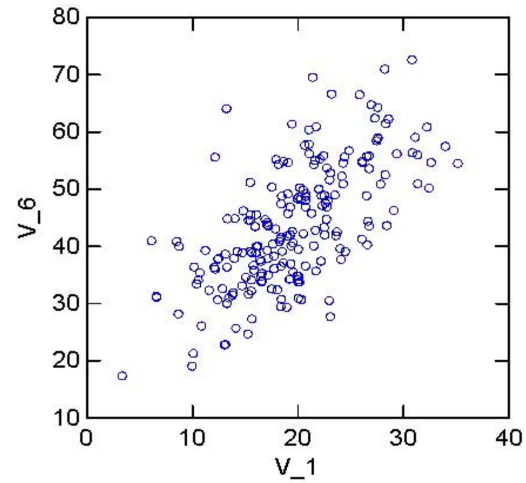
- Standardization of observed scores is not desired because invariance is a raw score regression problem.
- If standardization is desired, it must be done *using the same mean and standard deviation* for all occasions;

$$z[1]_n = (y[1]_n - \mu) / \sigma$$
$$z[2]_n = (y[2]_n - \mu) / \sigma$$

- This mean and standard deviation could come from, say, the time one scores for each manifest variable
- But, as in regression, questions about “equality of coefficients” cannot be answered if manifest variables are standardized within each occasion because information has been lost.
- Separate standardization violates the likelihood equations, so tests of fit are incorrect

4. An Example of Factorial Invariance over Time

WISC Verbal scores at two occasions (Grades 1 and 6; N=204)



Descriptive Statistics at T=2 Occasions 4 variables at each occasion

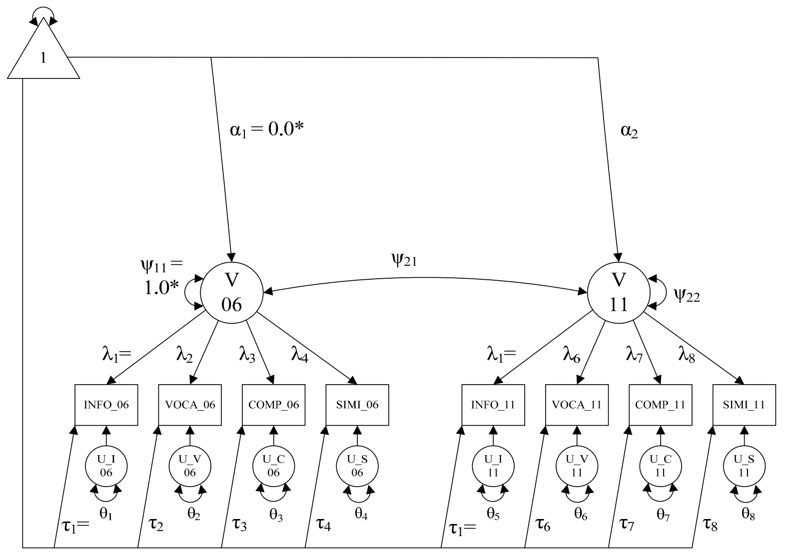
Variable	N	Mean	Std Dev
info6	204	19.77624	6.11922
comp6	204	21.79683	9.74162
simi6	204	14.90326	7.56052
voca6	204	20.39630	6.29161
info11	204	48.50969	12.78670
comp11	204	45.17377	12.97263
simi11	204	41.29651	14.51801
voca11	204	44.44644	11.04631

WISC correlations at T=2 Occasions

Pearson Correlation Coefficients, N = 204

	info6	comp6	simi6	voca6	info11	comp11	simi11	voca11
info6	1.000							
comp6	0.509	1.000						
simi6	0.540	0.449	1.000					
voca6	0.556	0.584	0.437	1.000				
info11	0.468	0.398	0.353	0.552	1.000			
comp11	0.363	0.356	0.295	0.443	0.624	1.000		
simi11	0.439	0.399	0.334	0.541	0.672	0.533	1.000	
voca11	0.485	0.463	0.380	0.598	0.749	0.701	0.665	1.000

Longitudinal Configural Invariance model



Partial Mplus code for Configural Invariance

```
          USEVAR = info_06 voca_06 comp_06 simi_06
                   info_11 voca_11 comp_11 simi_11;
ANALYSIS: TYPE=MEANSTRUCTURE; ITERATIONS=10000;

MODEL:   !Loadings on Latent Variables (L = loadings)
          Verb_06 BY info_06*(L1)
                     voca_06 (L2)
                     comp_06 (L3)
                     simi_06 (L4) ;
          Verb_11 BY info_11*(L1)
                     voca_11 (L6)
                     comp_11 (L7)
                     simi_11 (L8) ;
         !Intercepts (or taus) (I = intercepts)
          [info_06 info_11](I1);  [voca_06 voca_11];
          [comp_06 comp_11];      [simi_06 simi_11]    ;
         !Unique variances and covariances (U = Unique vars)
           info_06; info_11;      voca_06; voca_11;
           comp_06; comp_11;      simi_06; simi_11;
         !Latent variable means (A = alpha)
          [Verb_06@0]; [Verb_11];
         !Latent variable variances and covariances (P = psi)
           Verb_06@1;   Verb_11 ;
         !Regressions among latent variables (B = beta)
```


Configural Invariance, plus unique covs

```
MODEL:   !Loadings on Latent Variables (L = loadings)
          Verb_06 BY info_06*(L1)
                     voca_06 (L2)
                     comp_06 (L3)
                     simi_06 (L4) ;
          Verb_11 BY info_11*(L1)
                     voca_11 (L6)
                     comp_11 (L7)
                     simi_11 (L8) ;
         !Intercepts (or taus) (I = intercepts)
         [info_06 info_11](I1);  [voca_06 voca_11];
         [comp_06 comp_11];      [simi_06 simi_11]    ;
         !Unique variances and covariances (U = Unique vars)
          info_06; info_11;      voca_06; voca_11;
          comp_06; comp_11;      simi_06; simi_11;

          info_06 with info_11;  voca_06 with voca_11;
          comp_06 with comp_11;  simi_06 with simi_11;
         !Latent variable means (A = alpha)
         [Verb_06@0]; [Verb_11];
         !Latent variable variances and covariances (P = psi)
          Verb_06@1;   Verb_11 ;
         !Regressions among latent variables (B = beta)
```


Partial Mplus code for WEAK FI

```
MODEL:   !Loadings on Latent Variables (L = loadings)
          Verb_06 BY info_06*(L1)
                     voca_06 (L2)
                     comp_06 (L3)
                     simi_06 (L4) ;
          Verb_11 BY info_11*(L1)
                     voca_11 (L2)
                     comp_11 (L3)
                     simi_11 (L4) ;
         !Intercepts (or taus) (I = intercepts)
         [info_06 info_11](I1);  [voca_06 voca_11];
         [comp_06 comp_11];      [simi_06 simi_11]    ;
         !Unique variances and covariances (U = Unique vars)
          info_06; info_11;      voca_06; voca_11;
          comp_06; comp_11;      simi_06; simi_11;
         !Latent variable means (A = alpha)
         [Verb_06@0]; [Verb_11];
         !Latent variable variances and covariances (P = psi)
          Verb_06@1;   Verb_11;
         !Regressions among latent variables (B = beta)
```

Partial Mplus code for STRONG FI

```
MODEL:   !Loadings on Latent Variables (L = loadings)
          Verb_06 BY info_06*(L1)
                     voca_06 (L2)
                     comp_06 (L3)
                     simi_06 (L4) ;
          Verb_11 BY info_11*(L1)
                     voca_11 (L2)
                     comp_11 (L3)
                     simi_11 (L4) ;
         !Intercepts (or taus) (I = intercepts)
         [info_06 info_11](I1);  [voca_06 voca_11](I2);
         [comp_06 comp_11](I3);  [simi_06 simi_11](I4);
         !Unique variances and covariances (U = Unique vars)
          info_06; info_11;      voca_06; voca_11;
          comp_06; comp_11;      simi_06; simi_11;
         !Latent variable means (A = alpha)
         [Verb_06@0]; [Verb_11];
         !Latent variable variances and covariances (P = psi)
          Verb_06@1;   Verb_11;
         !Regressions among latent variables (B = beta)
```


Partial Mplus code for STRICT FI

```
MODEL:   !Loadings on Latent Variables (L = loadings)
          Verb_06 BY info_06*(L1)
                     voca_06 (L2)
                     comp_06 (L3)
                     simi_06 (L4) ;
          Verb_11 BY info_11*(L1)
                     voca_11 (L2)
                     comp_11 (L3)
                     simi_11 (L4) ;
         !Intercepts (or taus) (I = intercepts)
         [info_06 info_11](I1);  [voca_06 voca_11](I2);
         [comp_06 comp_11](I3);  [simi_06 simi_11](I4);
         !Unique variances and covariances (U = Unique vars)
          info_06-info_11 (U1);   voca_06-voca_11 (U2);
          comp_06-comp_11 (U3);   simi_06-simi_11 (U4);
         !Latent variable means (A = alpha)
         [Verb_06@0]; [Verb_11];
         !Latent variable variances and covariances (P = psi)
          Verb_06@1;   Verb_11 ;
         !Regressions among latent variables (B = beta)
```


Alternative Repeated Factor SEMs

WISC Verbal N=204, T=2	Test	1 Factor	Config. Inv.	Weak FI	Strong FI	Strict FI
Factor Loadings Time 1 Λ[1]	IN	4.69	4.45=	4.93=	5.27=	5.24=
	VO	4.76	5.04	4.87=	4.55=	4.51=
	CO	7.08	6.85	5.17=	4.53=	4.52=
	SI	4.89	4.59	5.07=	4.96=	5.03=
Factor Loadings Time 2 Λ[2]	IN	20.9	4.45=	4.93=	5.27=	5.24=
	VO	44.4	4.10	4.87=	4.55=	4.51=
	CO	45.2	4.01	5.17=	4.53=	4.52=
	SI	41.3	4.55	5.07=	4.96=	5.03=
Factor 1 Var		1.00*	1.00*	1.00*	1.00*	1.00*
Factor 2 Var			5.83	4.24	4.56	4.63
Factor cov [cor]			1.84[.76]	1.56[.76]	1.61[.75]	1.64[.76]
Factor mean [G2]			6.46	5.82	5.34	5.34
χ^2 / df fit		955 / 15	26 / 19	42 / 22	54 / 25	127 / 29
$\Delta\chi^2$ / Δdf fit			929 / 4	-16 / 3	-12 / 3	-73 / 4
CFI / TLI		.000/-.27	.992/.98	.976/.969	.965/.961	.880/.884
RMSEA (CI)		.427 (.40-.45)	.042 (.00-.08)	.067 (.04-.10)	.075 (.04-.10)	.129 (.11-.15)

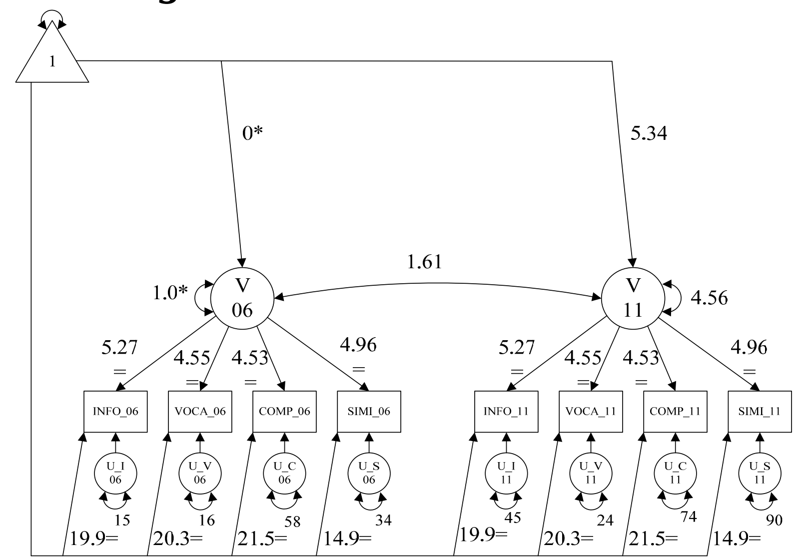
Testing Factor Invariance (FI)

- **Configural invariance:** baseline model
- **Weak FI:** Compare to configural invariance model to test invariance of factor loadings across time, $\Delta\chi^2 = 16$, $\Delta df = 3$
- **Strong FI:** Compare to Weak FI model to test invariance of intercepts across time, resulting in $\Delta\chi^2 = 12$, $\Delta df = 3$
- **Strict factorial invariance:** Compare to Strong FI model to test invariance of unique variances across time, $\Delta\chi^2 = 73$, $\Delta df = 4$

Conclusions from longitudinal WISC analyses

- A configuration based on a single factor for Verbal scales fits each time point reasonably well
- A Strong FI model with invariant loadings and intercepts was not perfect, but fit reasonably well
- Model of exactly equivalent factor scores (i.e., 1 factor) did not fit well.
- So, the same Verbal ability factor "grows" in **mean** and **variance** between ages 6 and 11.
- Growth in mean level (5.34) is a standardized effect size measure (i.e., Cohen's d) given scaling of factor at age 6

Longitudinal STRONG FI model



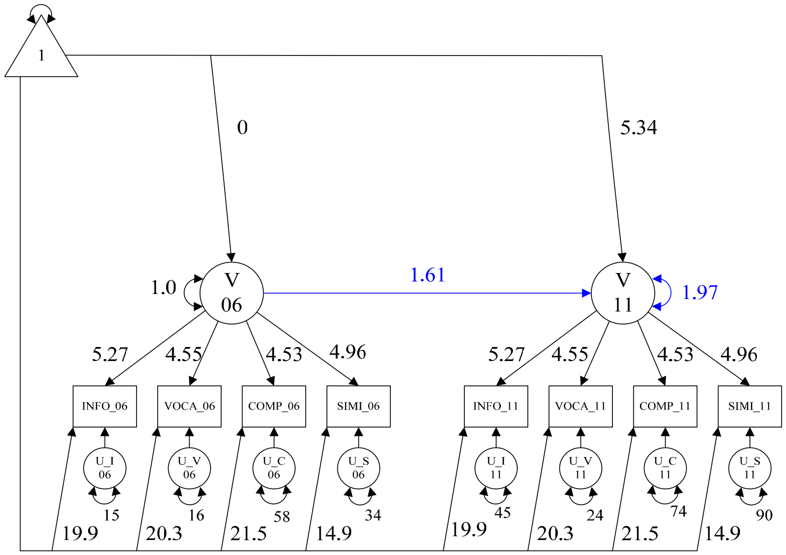
Two additional models for factor change

- Given reasonable fit of the Strong FI model, two additional "factor change" models can be specified
- Model 1: Autoregressive model at the factor level
- Model 2: Latent change in the latent factor
- Because both of these models have (a) the same number of estimates as the Strong FI model and (b) merely respecify the relations among latent variables, they have the same fit
- To specify and represent these models …

Mplus code for Autoregressive LV Model

```
MODEL:   !Loadings on Latent Variables (L = loadings)
          Verb_06 BY info_06*(L1)
                     voca_06 (L2)
                     comp_06 (L3)
                     simi_06 (L4) ;
          Verb_11 BY info_11*(L1)
                     voca_11 (L2)
                     comp_11 (L3)
                     simi_11 (L4) ;
         !Intercepts (or taus) (I = intercepts)
         [info_06 info_11](I1);  [voca_06 voca_11](I2);
         [comp_06 comp_11](I3);  [simi_06 simi_11](I4);
         !Unique variances and covariances (U = Unique vars)
          info_06; info_11;      voca_06; voca_11;
          comp_06; comp_11;      simi_06; simi_11;
         !Latent variable means (A = alpha)
         [Verb_06@0]; [Verb_11];
         !Latent variable variances and covariances (P = psi)
          Verb_06@1;   Verb_11 ;
          Verb_11 with Verb_06@0;
         !Regressions among latent variables (B = beta)
          Verb_11 on Verb_06;
```

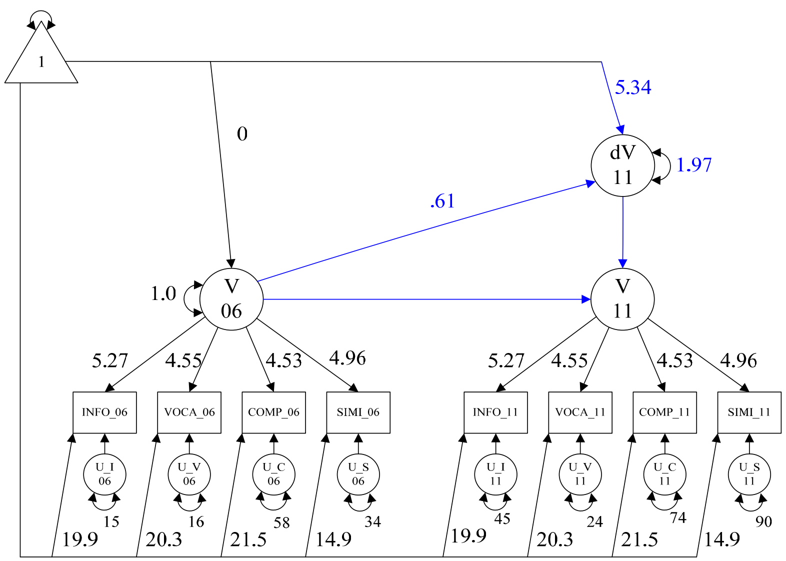
Longitudinal Autoregressive Model



Mplus code for Latent Change in LV Model

```
MODEL:   !Loadings on Latent Variables (L = loadings)
          Verb_06 BY info_06*(L1)
                     voca_06 (L2)
                     comp_06 (L3)
                     simi_06 (L4) ;
          Verb_11 BY info_11*(L1)
                     voca_11 (L2)
                     comp_11 (L3)
                     simi_11 (L4) ;
          dVer_11 BY Verb_11@1    ;
         !Intercepts (or taus) (I = intercepts)
         [info_06 info_11](I1);  [voca_06 voca_11](I2);
         [comp_06 comp_11](I3);  [simi_06 simi_11](I4);
         !Unique variances and covariances (U = Unique vars)
          info_06; info_11;      voca_06; voca_11;
          comp_06; comp_11;      simi_06; simi_11;
         !Latent variable means (A = alpha)
         [Verb_06@0]; [Verb_11@0];  [dVer_11];
         !Latent variable variances and covariances (P = psi)
          Verb_06@1;   Verb_11@0;    dVer_11;
          Verb_11 with Verb_06@0;
          Verb_11 with dVer_11@0;  Verb_06 with dVer_11@0;
         !Regressions among latent variables (B = beta)
          Verb_11 on Verb_06@1;    dVer_11 on Verb_06;
```


Latent Change In Latent Variable



6. Summary & Discussion

Additional Comments on Factorial Invariance

- The basic ideas of *factorial invariance* are a primary consideration in longitudinal SEM.
- Absence of strong factorial invariance makes it difficult:
  - to assert the same factors are measured with the same variables (i.e., "apples and oranges" problem)
  - to go any further with growth and change models (i.e., "rubber rulers").
- It seems reasonable for a researcher to examine various ways to achieve strong factorial invariance, even at the cost of factorial complexity (i.e., "life is not simple")

What if Strong FI does not hold?

- If Strong FI (or Strict FI) model does not fit, we might have evidence for a *qualitative* change, and this may be important in theory.
- Or, we might find that some specific variables may be the problem, and a model with only "partial invariance" in intercepts and/or loadings may be needed (this is a common solution nowadays).
- Alternatively we might consider some sampling theory of variables... and consider a more complex factor pattern (as in McArdle & Cattell, 1994).

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